The $\mathbf{d}^{++}$ Range Problem in Oriented Graphs

Abstract:

In an oriented graph D, the second out-degree of a vertex x, denoted by d^{++}(x) , is the number of vertices y such that d^{+}(x,y) = 2, where d^{+}(x,y) is the length of the shortest xy-directed path, if it exists. It is obvious that the sum of the first out-degrees of the vertices in an oriented graph is nothing but the number of its arcs. Unlike the first out-degree, the summation of the second out-degrees of the vertices, denoted by $\mathrm{d}^{++}(\mathrm{D}) = \sum_{v \in \mathrm{D}} \mathrm{d}^{++}(\mathrm{x})$, is not constant with respect to the number of vertices and arcs.

In [1], Alhussein and El Zein asked for the range of possible values of d^{++} (D) over all orientations D of a given undirected graph G, which we will call the d^{++}-range of G, and answered it for complete graphs, by characterizing, as a function of some integer n, the values that can be the summation of the second out-degrees of the vertices in a tournament of order n. In other words, they showed that for any integer $n \geq 6$, the summation of the second out-degrees in an n-tournament is i if and only if $i \in \{0, \ldots, n(n-1)/2\} \setminus \{1,2,4\}$. Moreover, they characterized n-tournaments whose summation of second out-degrees is $i \in \{0,3\}$. Then they asked for a characterization of n-tournaments having this summation i, $5 \leq i \leq n(n-1)/2$. Throughout the proof, they use the new concept of king degree in order to settle the problem. The king degree of a vertex x is the number of vertices that can be reached from x by a directed path of length at most 2.

Alhussein [2] answered the above question for $5 \leq i \leq 9$ and found the number of such tournaments in terms of n.

In [3], Alhussein and El Zein gave a complete description of the d^{++}- range for paths and cycles, and developed general tools for triangle-free graphs. As applications, they determined the exact range for grid graphs and established the extremal values of d^{++} (D) for trees through constructive proofs.

These results provide a foundation for further investigations of the d^{++}-range problem in oriented graphs, where several open problems are introduced.

References

[1] Aya Alhussein and Ayman El Zein, The king degree and the second out-degree of tournaments, *Discrete Mathematics* 348 (9) (2025) 114497.

[2] Aya Alhussein, Tournaments with fixed second out-degree summation: Characterization and enumeration, Under review.

[3] Aya Alhussein and Ayman El Zein, The d^{++} range problem in oriented graphs, Under review.